An Asymptotic Relation Between the Wirelength of an Embedding and the Wiener Index

K. Jagadeesh Kumar [a] Sandi Klavžar [b,c,d] R. Sundara Rajan [a]
Indra Rajasingh [e] T.M. Rajalaxmi [f]

[a] Department of Mathematics, Hindustan Institute of Technology and Science, Chennai-603 103, India
jagadeeshgraphs@gmail.com vprsundar@gmail.com

[b] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
sandi.klavzar@fmf.uni-lj.si

[c] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

[d] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

[e] School of Advanced Sciences, Vellore Institute of Technology, Chennai-600 127, India
indrarajasingh@yahoo.com

[f] Department of Mathematics, SSN College of Engineering, Chennai-603 110, India
laxmi.raji18@gmail.com

Abstract

Wirelength is an important criteria to validate the quality of an embedding of a graph into a host graph and is used in particular in VLSI layout designs. Wiener index plays a significant role in mathematical chemistry, cheminformatics, and elsewhere. In this note these two concepts are connected by proving that the Wiener index of a host graph in an upper bound for the wirelength of a given embedding. The wirelength of embedding complete 2^p-partite graphs into Cartesian products of paths and/or cycles as the function of the Wiener index is determined. The result is an asymptotic approximation of the general upper bound.

Keywords: Wiener index; embedding; wirelength; complete 2^p-partite graph; Cartesian product of graphs

1 Introduction

Given graphs G (guest) and H (host), an *embedding* of G into H is an injective mapping $f : V(G) \to V(H)$ together with an assignment that, to every edge $e = xy \in E(G)$, assigns a path $P_f(e)$ in H between $f(x)$ and $f(y)$. The *wirelength* [15] *of embedding G into H* is defined as

$$WL(G,H) = \min_{f:G\to H} \sum_{e=xy\in E(G)} m(P_f(e))\,,$$

where $m(P_f(e))$ is the number of edges of the path $P_f(e)$. The paths $P_f(e)$ in an embedding f of G into H in general need not be shortest paths because one can be interested in other properties of the embedding but the wirelength. On the other hand, $WL(G,H)$ will be realized on an embedding in which all paths $P_f(e)$ are shortest, hence the definition of the wirelength can be equivalently written as follows:

$$WL(G,H) = \min_{f:G\to H} \sum_{e=xy\in E(G)} d_H(f(x),f(y))\,,$$

where $d_G(u,v)$ denotes the length of a shortest path (that is, the number of its edges) between the vertices u and v of G. Wirelength is among the most important criteria to validate the quality of an embedding. It is used in particular in VLSI layout designs and has been well studied, see [1, 6, 11, 14, 15, 16].

The Wiener index (which is, for practical purposes, equivalent to the average distance) is an important concept in mathematics, computer science, and cheminformatics, to mention just some central areas of interest. For a graph G, the Wiener index $W(G)$ is defined as

$$W(G) = \frac{1}{2} \sum_{u\in V(G)} \sum_{v\in V(G)} d_G(u,v)\,.$$

This graph invariant is also used for understanding the biological phylogenetic diversity. In computer science, the average distance is used as a fundamental parameter to measure the communication cost of networks. For a very selected further information on the average distance we refer to [2, 13], and on the Wiener index to [2, 3, 4, 5, 8, 10, 18].

In the next section we connect the above described concepts by proving that the Wiener index of a host graph H in an upper bound for the wirelength of embedding G into H. In the subsequent section we derive formulas for the Wiener index of Cartesian products of a finite number of paths and/or cycles. In Section 4 we combine these expressions with earlier results to determine the wirelength of embedding complete 2^p-partite graphs into Cartesian products of paths and/or cycles as a function of the Wiener index. This results is an asymptotic approximation of the bound from Section 2. We conclude with examples demonstrating that embedding complete 2^p-partite graphs into some other host graphs does not have this property. To conclude the introduction we list some further definitions and concepts.

Graphs in this note are connected, unless stated otherwise. The order of a graph G is denoted with $n(G)$. The complete p-partite graph $G = K_{n_1,\ldots,n_p}$ is a graph that contains p independent sets with respective cardinalities n_i, $i \in [p] = \{1,\ldots,p\}$, and all possible edges between vertices from different parts. The *Cartesian product* $G\,\Box\,H$ of (not necessarily connected) graphs G and H is the graph with the vertex set $V(G)\times V(H)$, vertices (u,v) and (u',v') being adjacent if either $u=u'$ and $vv'\in E(H)$, or $v=v'$ and $uu'\in E(G)$. Networks generated by Cartesian product of networks are very powerful in creating a large network from given small graphs and it is an important technique for planning large-scale interconnection networks. For more information on Cartesian product graphs see the book [7].

2 The connection

If a graph G is sparser than a graph H, then $WL(G,H)$ is expected to be relatively small. For instance, if G is a spanning subgraph of H, then the identity mapping $V(G)\to V(H)$ obtained by considering G to be spanned in H, yields $WL(G,H)\le m(G)$. Since $m(G)$ is clearly a general lower bound for $WL(G,H)$, this means that $W(G,H) = m(G)$ provided that G is a spanning graph of H. On the other hand, if G is relatively dense, then we roughly expect that $WL(G,H)$ is relatively large. What large means is explained in the next result which connects the wirelength with the Wiener index.

Theorem 2.1. *If G and H are graphs with $n(G) = n(H)$, then $WL(G,H) \leq W(H)$. The equality holds if and only if G is a complete graph.*

Proof. Let f be a mapping $V(G) \to V(H)$ for which $WL(G,H)$ is realized, that is,

$$WL(G,H) = \sum_{xy \in E(G)} d_H(f(x), f(y))\,.$$

Since for every edge $xy \in E(G)$ we have $f(x) \neq f(y)$, and for every two different edges $xy, x'y' \in E(G)$ we also have $\{f(x), f(y)\} \neq \{f(x'), f(y')\}$, we can estimate as follow:

$$\begin{aligned} WL(G,H) &= \sum_{xy \in E(G)} d_H(f(x), f(y)) \\ &= \sum_{\substack{xy \in E(G) \\ \{f(x),f(y)\} \in \binom{V(H)}{2}}} d_H(f(x), f(y)) \\ &\leq \sum_{\{u,v\} \in \binom{V(H)}{2}} d_H(u,v) \\ &= W(H)\,. \end{aligned}$$

The equality in the inequality above holds if and only if the sets $\{f(x), f(y)\}$ run over all 2-subsets of $V(H)$. This holds if and only if G has $|\binom{V(H)}{2}| = |\binom{V(G)}{2}|$ edges, that is, if and only if G is a complete graph. $\square$

3 Wiener index of Cartesian products

The Wiener index of Cartesian products graphs has been independently obtained several times, the seminal paper being [5, 18]. The result says that if G and H are graphs, then

$$W(G \,\square\, H) = n(G)^2 \cdot W(H) + n(H)^2 \cdot W(G)\,. \tag{1}$$

The simplest way to obtain (1) is to apply the so-called Distance Lemma which asserts that if G and H are graphs, and $(g,h), (g',h')$ are vertices of $V(G \,\square\, H)$, then $d_{G \square H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h')$. From here the formula (1) follows by a straightforward computation.

Since the Cartesian product operation is associative, Distance Lemma naturally extends to more than two factors. More precisely, if $k \geq 2$ and $G = \square_{i=1}^{k} G_i$, where G_i, $i \in [k]$, are graphs, then

$$d_G(g,g') = \sum_{i=1}^{k} d_{G_i}(g_i, g_i')\,, \tag{2}$$

where $g = (g_1, \ldots, g_k)$ and $g' = (g_1', \ldots, g_k')$ are vertices of G. From here one can again more or less straightforwardly deduce the Wiener index of Cartesian products of a finite number of factors, cf. [9, p. 46].

Proposition 3.1. *If $k \geq 2$, and G_i, $i \in [k]$ are graphs, then*

$$W(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_k) = \sum_{i=1}^{k} \left(W(G_i) \cdot \prod_{j \neq i} n(G_j)^2 \right)\,.$$

The following consequences of Proposition 3.1 are needed for our purpose.

Corollary 3.2. *If $k \geq 2$ and $r_i \geq 2$, $i \in [k]$, are integers such that $r_1 + \cdots + r_k = r$, then the following hold.*

(i) $W(P_{2^{r_1}} \Box \cdots \Box P_{2^{r_k}}) = \frac{2^{2r}}{6}\left[(2^{r_1} + \cdots + 2^{r_k}) - \left(\frac{1}{2^{r_1}} + \cdots + \frac{1}{2^{r_k}}\right)\right]$.

(ii) $W(C_{2^{r_1}} \Box \cdots \Box C_{2^{r_k}}) = 2^{2r-3} \cdot (2^{r_1} + \cdots + 2^{r_k})$.

(iii) $W(P_{2^{r_1}} \Box \cdots \Box P_{2^{r_s}} \Box C_{2^{r_{s+1}}} \Box \cdots \Box C_{2^{r_k}}) = \frac{1}{6}\sum_{i=1}^{s} 2^{2r-r_i}(2^{2r_i} - 1) + \sum_{i=s+1}^{k} 2^{2r+r_i-3}$.

Proof. (i) It is well-known (and easy to see) that $W(P_n) = \binom{n+1}{3} = n(n^2-1)/6$. Combining this fact with Proposition 3.1 we get:

$$\begin{aligned} W(P_{2^{r_1}} \Box \cdots \Box P_{2^{r_k}}) &= \sum_{i=1}^{k} \frac{1}{6} 2^{r_i}\left(2^{2r_i} - 1\right) \cdot \prod_{j \neq i} 2^{2r_j} = \frac{1}{6}\sum_{i=1}^{k} 2^{r_i}\left(2^{2r_i} - 1\right) \cdot \frac{2^{2r}}{2^{2r_i}} \\ &= \frac{2^{2r}}{6}\sum_{i=1}^{k}\left(2^{2r_i} - 1\right) \cdot \frac{1}{2^{r_i}} \\ &= \frac{2^{2r}}{6}\sum_{i=1}^{k}\left(2^{r_i} - \frac{1}{2^{r_i}}\right). \end{aligned}$$

Proof for (ii) proceed along the same lines as for (i). The formula (iii) is then obtained by using the associativity of the Cartesian product and writing

$$P_{2^{r_1}} \Box \cdots \Box P_{2^{r_s}} \Box C_{2^{r_{s+1}}} \Box \cdots \Box C_{2^{r_k}} = (P_{2^{r_1}} \Box \cdots \Box P_{2^{r_s}}) \Box (C_{2^{r_{s+1}}} \Box \cdots \Box C_{2^{r_k}}),$$

and then applying (1) together with the already established formulas (i) and (ii). □

4 Asymptotically largest possible wirelengths

In this section we prove an exact formula for the wirelength as a function of the Wiener index that can be viewed as an asymptotic approximation of the bound from Theorem 2.1.

Theorem 4.1. *Let G be the complete 2^p-partite graphs $K_{2^{r-p},2^{r-p},\ldots,2^{r-p}}$, where $p \geq 1$, $r \geq 3$ and $p < r$. Let H be the Cartesian product of $k \geq 3$ factors of respective order 2^{r_i}, $i \in [k]$, where $r_1 + \cdots + r_k = r$, and each factor is a path or a cycle. Then*

$$WL(G, H) = \frac{(2^p - 1)}{2^p}\, W(H).$$

Proof. If G, p, r, H, and k are as stated, then it was proved in [17] the following. If $s \geq 0$ factors of H are paths and the other factors are cycles, then,

$$WL(G, H) = \frac{1}{6}\sum_{i=1}^{s} 2^{2r-r_i-p}(2^p - 1)(2^{2r_i} - 1) + \sum_{i=s+1}^{k} 2^{2r+r_i-p-3}(2^p - 1).$$

(Note that this formula also includes the cases when all the factors are paths ($s = k$) and when all the factors are cycles ($s = 0$).) The result then follows by comparing the above formula with Corollary 3.2. □

A question arises, whether the equality $WL(G, H) = \frac{(2^p-1)}{2^p} W(H)$ can hold for some additional host graphs H with $n(G) = n(H)$, where G is the complete 2^p-partite graph $K_{2^{r-p},2^{r-p},\ldots,2^{r-p}}$, $r \geq 3$, $p \geq 1$ and $p < r$. This is not the case in the following two examples.

Let $G = K_{8,8,8,8}$ and $H = P_{32}$. It is straightforward to check that $W(H) = 5456$. On the other hand, from [16] we have $WL(G, H) = 4112$. Hence,

$$WL(G, H) = 4112 > \frac{(2^p - 1)}{2^p} W(H) = \frac{3}{4}(5456) = 4092\,.$$

In the second example let $G = K_{4,4,4,4}$ and let H be the circulant graph $G(16; \pm\{1, 2\})$ as shown in Fig. 1.

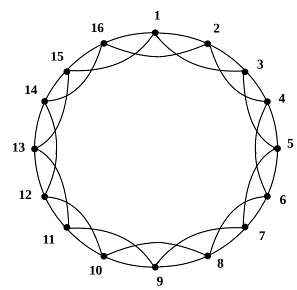

Figure 1: The circulant graph $G(16; \pm\{1, 2\})$

It is easy to verify that $W(H) = 320$, while from [16] we have $WL(G, H) = 216$. Hence,

$$WL(G, H) = 216 < \frac{(2^p - 1)}{2^p} W(H) = \frac{3}{4}(320) = 240\,.$$

These observations lead to:

Problem 1. Find families of (host) graphs H such that

$$WL(G, H) = \frac{(2^p - 1)}{2^p} W(H)$$

holds, where G is the complete 2^p-partite graph $K_{2^{r-p},\ldots,2^{r-p}}$, $r \geq 3$, $p \geq 1$ and $p < r$.

5 Conclusion

In this note we have obtained the wirelength $WL(G, H)$ of embedding G onto H using the Wiener index of H, where G is the complete 2^p-partite graph $K_{2^{r-p},\ldots,2^{r-p}}$ and H is the Cartesian product of paths and cycles. Finding the wirelength of embedding complete multipartite graph into graphs such as Cayley graphs, permutation graphs, and interval graphs are under investigation.

Acknowledgments

We thank Ms. A. Arul Shantrinal, Department of Mathematics and Ms. G. Kirithiga Nandini, Department of Computer Science and Engineering, Hindustan Institute of Technology and Science, Chennai, India, for their fruitful suggestions.

R. Sundara Rajan was supported by Project No. ECR/2016/1993, Science and Engineering Research Board (SERB), Department of Science and Technology (DST), Government of India.

Sandi Klavžar was supported from the Slovenian Research Agency (research core funding No. P1-0297 and projects J1-9109, J1-1693, N1-0095).

References

[1] S.L. Bezrukov and U.-P. Schroeder, The cyclic wirelength of trees, Discrete Appl. Math. 87 (1998) 275–277.

[2] R. M. Casablanca, Average distance in the strong product of graphs, Util. Math. 94 (2014) 31–48.

[3] M. Eliasi, G. Raeisi and B. Taeri, Wiener index of some graph operations, Discrete Appl. Math. 160 (2012) 1333–1344.

[4] M. Goubko, Minimizing Wiener index for vertex-weighted trees with given weight and degree sequences, MATCH Commun. Math. Comput. Chem. 75 (2016) 3–27.

[5] A. Graovac, T. Pisanski, On the Wiener index of a graph, J. Math. Chem. 8 (1991) 53–62.

[6] C. J. Guu, The Circular Wirelength Problem for Hypercubes, Ph.D. dissertation, University of California, Riverside, (1997).

[7] R. Hammack, W. Imrich, S. Klavžar, Handbook of product graphs, Second Edition, CRC Press, Boca Raton, FL, (2011).

[8] A. Ilić, A note on 'A new approach to compute Wiener index', J. Comput. Theor. Nanosci. 11 (2014) 958–960.

[9] S. Klavžar, A. Rajapakse, I. Gutman, The Szeged and the Wiener index of graphs, Appl. Math. Lett. 9 (1996) 45–49.

[10] P. Manuel, I. Rajasingh, B. Rajan, R.S. Rajan, A new approach to compute Wiener index, J. Comput. Theor. Nanosci. 10 (2013) 1515–1521.

[11] P. Manuel, I. Rajasingh, B. Rajan and H. Mercy, Exact wirelength of hypercube on a grid, Discrete Appl. Math. 157 (2009) 1486–1495.

[12] B. Mohar, T. Pisanski, How to compute the Wiener index of a graph, J. Math. Chem. 2 (1988) 267–277.

[13] S. Mukwembi, Average distance, independence number and spanning trees, J. Graph Theory 76 (2014) 194–199.

[14] I. Rajasingh, M. Arockiaraj, B. Rajan and P. Manuel, Minimum wirelength of hypercubes into n-dimensional grid networks, Inform. Process. Lett. 112 (2012) 583–586.

[15] I. Rajasingh and R.S. Rajan, Exact wirelength of embedding circulant networks into necklace and windmill Graphs, Ars Combin. 130 (2017) 215–237.

[16] R.S. Rajan, T.M. Rajalaxmi, J.B. Liu and G. Sethuraman, Wirelength of embedding complete multipartite graphs into certain graphs, Discrete Appl. Math. (2018), https://doi.org/10.1016/j.dam.2018.05.034.

[17] R.S. Rajan, A.A. Shantrinal, K.J. Kumar, T.M. Rajalaxmi, J. Fan, and W. Fan, Embedding complete multi-partite graphs into cartesian product of paths and cycles, submitted.

[18] Y.-N. Yeh, I. Gutman, On the sum of all distances in composite graphs, Discrete Math. 135 (1994) 359–365.